* Runge - Kutta Method

In this method, the derivatives of higher order are not required and we require only the given function values at different points.

Second Order Runge Kutta method

Aim: To solve $\frac{dy}{dx} = f(x, y)$ given $y(x_0) = y_0$. $\rightarrow (1)$

Proof

By Taylor series, we have

$$y(x+h) = y(x) + h\, y'(x) + \frac{h^2}{2!} y''(x) + O(h^3) \rightarrow (2)$$

Differentiating (1) w.r.t x

$$y'' = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dx} = f_x + y' f_y = f_x + f f_y$$

$\rightarrow (3)$

Using the values of y' and y'' got from (1) and (3), in (2), we get

$$y(x+h) - y(x) = hf + \frac{1}{2} h^2 [f_x + f f_y] + O(h^3)$$

$$\therefore \Delta y = hf + \frac{1}{2} h^2 (f_x + f f_y) + O(h^3) \rightarrow (4)$$

Let $\Delta_1 y = k_1 = f(x, y) . \Delta x = h f(x, y) \rightarrow (5)$

$$\Delta_2 y = k_2 = h\,f(x+mh,\ y+mk_1) \rightarrow (6)$$

and let $\Delta y = ak_1 + bk_2 \rightarrow (7)$

where a, b and m are constants to be determined to get the better accuracy of Δy.

Expand k_2 and Δy in powers of h.

Expanding k_2, by Taylor Series for two variables, we have

$$k_2 = hf(x+mh,\ y+mk_1)$$

$$= h\left[f(x,y) + \left(mh\frac{\partial}{\partial x} + mk_1\frac{\partial}{\partial y}\right)f + \frac{\left(mh\frac{\partial}{\partial x} + mk_1\frac{\partial}{\partial y}\right)^2}{2!}f + \cdots\right]$$

$$= h\left[f + mhf_x + mhff_y + \frac{\left(mh\frac{\partial}{\partial x} + mk_1\frac{\partial}{\partial y}\right)^2 f}{2!} + \cdots\right.$$

since $k_1 = hf$ $\rightarrow (8)$

$$= hf + mh^2(f_x + ff_y) + \cdots \text{ highest powers of } h. \rightarrow (9)$$

Substituting k_1, k_2 in (7),

$$\Delta y = ahf + b\left[hf + mh^2(f_x + ff_y) + O(h^3)\right]$$

$$= (a+b)hf + bmh^2(f_x + ff_y) + O(h^3) \rightarrow (10)$$

Equating Δy from (4) and (10), we get

$$a+b=1 \text{ and } bm = \frac{1}{2} \rightarrow (11)$$

Now we have only two equations given by (1) to solve for three unknowns a, b, m.

From $a+b=1$, $a = 1-b$ and also

$$m = \frac{1}{2b} \text{ using (7),}$$

$$\Delta y = (1-b)k_1 + bk_2$$

where $k_1 = hf(x,y)$

$$k_2 = hf\left(x+\frac{h}{2b}, y+\frac{hf}{2b}\right)$$

Now, $\Delta y = y(x+h) - y(x)$

$$\therefore y(x+h) = y(x) + (1-b)hf + bhf\left(x+\frac{h}{2b}, y+\frac{hf}{2b}\right)$$

i.e.,
$$y_{n+1} = y_n + (1-b)hf(x_n, y_n) + bhf\left(x_n+\frac{h}{2b}, y_n+\frac{h}{2b}f(x_n,y_n)\right) + O(h^3).$$

From this general second order Runge-kutta formula, setting $a=0$, $b=1$, $m=\frac{1}{2}$, we get

$$\boxed{\begin{aligned} k_1 &= hf(x,y) \\ k_2 &= hf\left(x+\tfrac{1}{2}h, y+\tfrac{1}{2}k_1\right) \\ k_3 &= hf(x+h, y+2k_2-k_1) \\ \text{and } \Delta y &= \tfrac{1}{6}(k_1+4k_2+k_3) \end{aligned}}$$

This is second order Runge-kutta algorithms.

Problem

4. Obtain the values of y at $x = 0.1, 0.2$ using R.K. method of second order for the differential equation $y' = -y$, given $y(0) = 1$.

Sol

Here $f(x,y) = -y$, $x_0 = 0$, $y_0 = 1$, $x_1 = 0.1$,
$x_2 = 0.2$

second order:

$$k_1 = hf(x_0, y_0) = (0.1)(-y_0) = -0.1$$

$$k_2 = hf\left(x_0 + \tfrac{1}{2}h,\ y_0 + \tfrac{1}{2}k_1\right) = (0.1)\,f(0.05, 0.95)$$

$$= -0.1 \times 0.95$$

$$= -0.095 = \Delta y.$$

$$y_1 = y_0 + \Delta y = 1 - 0.095 = 0.905$$

$$y_1 = y(0.1) = 0.905.$$

Again starting from $(0.1, 0.905)$ replacing (x_0, y_0) by (x_1, y_1) we get

$$k_1 = (0.1)f(x_1, y_1) = (0.1)(-0.905) = -0.0905$$

$$k_2 = hf\left(x_1 + \tfrac{1}{2}h,\ y_1 + \tfrac{1}{2}k_1\right)$$

$$= (0.1)[f(0.15, 0.859775)] = (0.1)(-0.859775)$$

$$= -0.0859775$$

$$\Delta y = k_2$$

$$\therefore\ y_2 = y(0.2) = y_1 + \Delta y = 0.819025.$$